Sub-Cellular Technique for Finite-Difference Time-Domain Method

Sunil Kapoor, *Member, IEEE*

***Abstract*—The most frequently encountered problem with the finite-difference time-domain (FDTD) method, for the analysis of microstrip line and many other structures, is that the structure generally has important structural features that are very small as compared to the main body in at least one dimension. Thus, to accurately analyze such structures, very small mesh is considered which makes the analysis very expensive, time consuming, and sometimes even impossible due to the computer limitations. In this paper, a new sub-cellular technique has been proposed which takes care of such problems as well as many other problems, such as curved surfaces (in which interfaces are not parallel to one of the coordinate planes and stair-step approximation is considered).**

***Index Terms*— Electromagnetic, finite-difference time-domain (FEDTD), mathematical technique, microstrip line, sub-cellular.**

I. Introduction

THE time-dependent Maxwell's differential equations can be represented by a set of difference equations and can be solved numerically on a computer. This method of solving the Maxwell's differential equations is popularly known as the finite-difference time-domain (FDTD) method and was first proposed in [1] in two dimensions and later applied to three dimensions in [2]–[11]. The FDTD method of solving Maxwell's equations is becoming more and more popular due to its simplicity.

There are still some structures for which the application of the FDTD method is not preferred—for example, the structures in which some part has very small dimension in at least one direction and, therefore, to analyze it accurately, very small mesh has to be considered. Reducing the cell size throughout the FDTD computational space is one method for dealing with this type of situation, but it is computationally very expensive and may not even be practical if computer resources, such as memory and speed, are inadequate. Also, some structures have a curvilinear boundary which needs stair-step approximation and, thus, results in loss of accuracy. In such cases, accuracy is proportional to the cell dimensions and usually requires small cells if the curve is sharp, which again poses the same problems of expensive computation, memory, and computer limitations.

Few methods have been proposed so far [12]–[21], which are being used extensively to overcome (to some extent) such problems. Some of the methods [12], [15] use large FDTD cells throughout the computation space but approximate the small geometry elements by modifying the equations for the large cells that contain them. For example, a surface impedance concept may be used to include material layers thinner than the FDTD cells [12], [13]. Another variation involves special equations for calculating the fields in the vicinity of discontinuity thinner than the FDTD cell size [14], [15]. Effects of lumped circuit elements which are contained within one FDTD cell may also be included by modifying the field equations for that cell. Development of this approach often involves application of Maxwell's equations in integral rather than differential form, but the finite-difference equations can be obtained from the integral form of Maxwell's equation. These methods are complex, as they need modification of the equations.

One of the most commonly used methods to deal with such geometries is the expansion technique [16], [17]. In the expansion technique, the region of interest in the object being analyzed is replaced with a finer grid. This method allows realistic interior response predictions to be made. The technique consists of making an initial computer run with a model of the entire system. The electric fields, scattered from the system and tangential to a sub-boundary, are stored on disk from this calculation. The portion of the system inside the sub-boundary is then subdivided into smaller cells and the sub-boundary becomes the outer boundary for a second calculation. The same incident field used for the first calculation illuminates the subdivided portion of the system on the second calculation. The same tangential E-field response as seen on the sub-boundary for the first run is imposed on the outer boundary of the second run. The advantage of the second run with smaller grid cells is that the missing portion of the system appears to be present at this time—at least at low frequencies. The drawback of this method is that it is very slow as far as processing speed is concerned. The computer takes a long time to store and then read the data from the disk every time the mesh size is changed.

Some of the other methods include the sub-cell model [18] and the contour path (CP) method [19]. The sub-cell model is basically helpful in analyzing thin resistive films used in waveguide components, substrate-mounted thin metallic or dielectric films used in integrated circuitry, thin dielectric windows or radomes used to enclose antennas, etc. In the sub-cell model, the basic Cartesian grid arrangement of field components at all space cells, except those special cells containing the thin material sheet, are preserved. In special cells, the electric-field component normal to the sheet is split in two parts. The tangential components of the field do not

Manuscript received May 1, 1996; revised November 6, 1996.

The author is with Ward Products Corporation, New Brunswick, NJ 08903 USA.

Publisher Item Identifier S 0018-9480(97)02908-6.

need to be split as they are continuous across the boundary. The equations for the special cells are updated from the integral form of Maxwell's equations. The line integrals are performed assuming that the field is constant over segments of the contour, and the surface integrals are performed assuming the field as constant over the area of integration. Finally, the time derivative is simply approximated by a finite difference. The drawback of this method is that it has a very limited validity range.

In the CP method [19], which is basically used for curvilinear boundaries, the basic Cartesian grid arrangement of field components at all space cells, except those immediately adjacent to the structure surface, are preserved. Space cells adjacent to the structure surface are deformed to conform with the surface locus. Slightly modified time-stepping expressions for the field components adjacent to the surface are obtained by applying the CP technique. The drawback of this method is that it has a very restricted validity range.

Some other proposed methods are the variable step size method (VSSM) [20] and the mesh refinement algorithm (MRA) [21], which are similar to the proposed sub-cellular technique in the sense that they also make use of dividing the computational domain into two or more regions with different cell size. The disadvantage in using these methods is that both require calculation of extra second-order difference equations at each coarse node on the boundary and also memorize extra second-order differences—the required number of memorizing fields increases with the increase in the reduction ratio. The difference between the MRA and VSSM is that in the VSSM, the second-order differences are calculated from spatially interpolated field values, whereas in the MRA, the second-order differences are calculated and then interpolated in space.

In this paper, a new method has been proposed which is much simpler compared to some of the methods explained above, as no modifications in the field equations are required. This proposed method is much faster, as it helps in reducing the number of cells and there is no requirement of storing/reading the fields on/from the disk. According to this method, the system is divided into two or more regions and different cell sizes are considered in each region. The fields in each region are calculated using the conventional FDTD method where as fields on and around the boundary (between the regions of different cell sizes) are calculated using interpolation technique as explained below in the 15-point procedure.

II. Implementation

The finite-difference system equations for electric (E) and magnetic (H) fields in ($i,\ j,\ k$)th cell can be written as

$$\begin{aligned}
E_x(i,j,k)|^{n+1} &= A.E_x(i,j,k)|^n \\
&\quad + B.\{H_z(i,j+0.5,k)|^{n+0.5} \\
&\quad - H_z(i,j-0.5,k)|^{n+0.5} \\
&\quad + H_y(i,j,k-0.5)|^{n+0.5} \\
&\quad - H_y(i,j,k+0.5)|^{n+0.5}\} \\
E_y(i,j,k)|^{n+1} &= A.E_y(i,j,k)|^n \\
&\quad + B.\{H_x(i,j,k+0.5)|^{n+0.5} \\
&\quad - H_x(i,j,k-0.5)|^{n+0.5} \\
&\quad + H_z(i-0.5,j,k)|^{n+0.5} \\
&\quad - H_z(i+0.5,j,k)|^{n+0.5}\} \\
E_z(i,j,k)|^{n+1} &= A.E_z(i,j,k)|^n \\
&\quad + B.\{H_y(i+0.5,j,k)|^{n+0.5} \\
&\quad - H_y(i-0.5,j,k)|^{n+0.5} \\
&\quad + H_x(i,j-0.5,k)|^{n+0.5} \\
&\quad - H_x(i,j+0.5,k)|^{n+0.5}\} \\
H_x(i,j,k)|^{n+0.5} &= C.H_x(i,j,k)|^{n-0.5} \\
&\quad + D.\{E_z(i,j-0.5,k)|^n \\
&\quad - E_z(i,j+0.5,k)|^n \\
&\quad + E_y(i,j,k+0.5)|^n \\
&\quad - E_y(i,j,k-0.5)|^n\} \\
H_y(i,j,k)|^{n+0.5} &= C.H_y(i,j,k)|^{n-0.5} \\
&\quad + D.\{E_z(i+0.5,j,k)|^n \\
&\quad - E_z(i-0.5,j,k)|^n \\
&\quad + E_x(i,j,k-0.5)|^n \\
&\quad - E_x(i,j,k+0.5)|^n\} \\
H_z(i,j,k)|^{n+0.5} &= C.H_z(i,j,k)|^{n-0.5} \\
&\quad + D.\{E_x(i,j+0.5,k)|^n \\
&\quad - E_x(i,j-0.5,k)|^n \\
&\quad + E_y(i-0.5,j,k)|^n \\
&\quad - E_y(i+0.5,j,k)|^n\}
\end{aligned}$$

where

$$A = \frac{1-\dfrac{\sigma_{i,j,k}\Delta t}{2\varepsilon_{i,j,k}}}{1+\dfrac{\sigma_{i,j,k}\Delta t}{2\varepsilon_{i,j,k}}}$$

$$B = \frac{\dfrac{\Delta t}{\varepsilon_{i,j,k}\Delta}}{1+\dfrac{\sigma_{i,j,k}\Delta t}{2\varepsilon_{i,j,k}}}$$

$$C = \frac{1-\dfrac{\rho_{i,j,k}\Delta t}{2\mu_{i,j,k}}}{1+\dfrac{\rho_{i,j,k}\Delta t}{2\mu_{i,j,k}}}$$

$$D = \frac{\dfrac{\Delta t}{\mu_{i,j,k}\Delta}}{1+\dfrac{\rho_{i,j,k}\Delta t}{2\mu_{i,j,k}}}.$$

The subscript $x,\ y$, and z represents the x-, y-, and z-directed fields and superscript n represents the fields at the nth time. σ, $\rho,\ \varepsilon$, and μ are the electric conductivity, magnetic resistivity, electric permittivity, and magnetic permeability constants of the ($i,\ j,\ k$)th cell.

In the case of sub-cellular technique, the computational domain is divided into two or more regions, with each re-

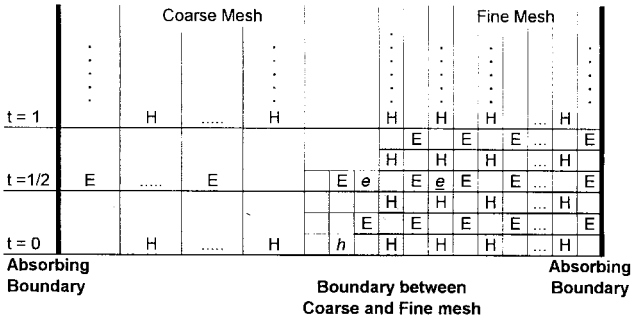

Fig. 1. Implementation of sub-cellular technique in the FDTD method.

gion having a different cell size. Fields are calculated using the finite-difference system equations. At some points, in different time and space iterations, the fields are unknown and are required for calculating other fields in the next time iteration. Following 15-step iterative procedure has been used for calculating these fields. The procedure has been explained using Fig. 1, and here, only two regions have been considered—the coarse-grid and fine-grid regions (three fine-grid cells = one coarse-grid cell case). In Fig. 1, $\mathbf{E}$ and $\mathbf{H}$ represents the electric and magnetic fields where as $\mathbf{t}$ represents the time-stepping. Here, centered finite-difference expressions for the space and time derivatives (using the central-difference method) have been used, which are second-order accurate in space and time. Also, the leapfrog time-stepping process has been implemented as was suggested in [1]. In Fig. 1, $\boldsymbol{e}$ and $\underline{\boldsymbol{e}}$ represent the electric fields calculated using interpolation and, similarly, $\boldsymbol{h}$ is the magnetic field calculated using interpolation as explained below:

1) update H-fields on the coarse side;
2) update H-fields on the fine side;
3) interpolate H-field on the coarse side, shown as h in Fig. 1, using immediate H-fields on both the sides as shown below:

$$h = 0.4\,H_c + 0.6\,H_b \tag{1}$$

where H_c is the H-field on the coarse-grid side just before the boundary and H_b is the H-field on the boundary. The calculation of weighting coefficients of H_c and H_b for the general case has been given in the Appendix.
4) update E-field on coarse side (just before the boundary), and all the E-fields on the fine side;
5) update H-fields on the fine side and on the boundary;
6) update E-fields on the coarse side;
7) update E-fields on the fine side;
8) interpolate E-fields, shown as e and $\underline{e}$ in Fig. 1, using the immediate E-fields on both sides as given below:

$$e = 0.67\,E_c + 0.33\,E_f \tag{2}$$

where E_c is the E-field on the coarse side just before the boundary and E_f is the E-field on the fine side just after the boundary. Again, the calculation of weighting coefficients of E_c and E_f for the general case has been given in the Appendix. Similarly

$$\underline{e} = 0.5\,(E_{fl} + E_{fr})$$

where E_{fl} and E_{fr} are the E-fields on the immediate left and right side in the fine-grid area, respectively;
9) update H-field on the boundary using the interpolated e field and first updated E-field on the fine side;
10) update rest of H-fields on the fine side using the updated E-fields;
11) update E-fields on the fine side;
12) update H-fields on the coarse side;
13) update H-field on the boundary, between coarse and fine mesh, using E-fields just before the boundary and the interpolated $\underline{e}$ field;
14) update H-fields on the fine side;
15) repeat step 3 onwards.

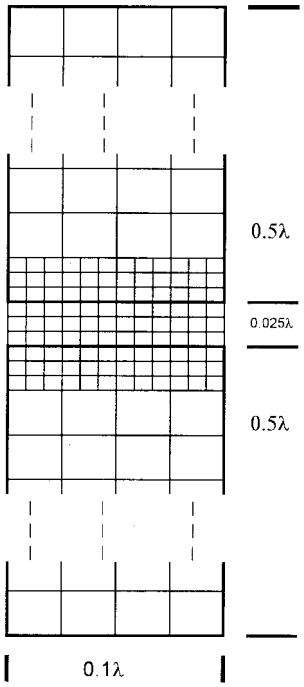

Fig. 2. Mesh distribution shown for a conducting screen with a thin slot.

It is essential to check the stability condition in a finite-difference method which guarantees that the numerical error generated in one step of the calculation does not accumulate and grow. In Yee's algorithm [1], which has been followed here, the stability condition is

$$v_{\max}.\,\Delta t \le \left[(\Delta x)^{-2} + (\Delta y)^{-2} + (\Delta z)^{-2}\right]^{-0.5}$$

where $v_{\max}$ is the maximum phase velocity of the signal and Δx, Δy, and Δz are the dimensions of a coarse grid cell in the x-, y-, and z-directions, respectively. For the special case of $\Delta x = \Delta y = \Delta z = \Delta h$, the above equation becomes

$$v_{\max} \cdot \Delta t \le \frac{\Delta h}{\sqrt{3}}.$$

The stability of the absorbing boundary conditions cannot be achieved exactly as all the presently available absorbing boundary conditions are imperfect for the numerical solution

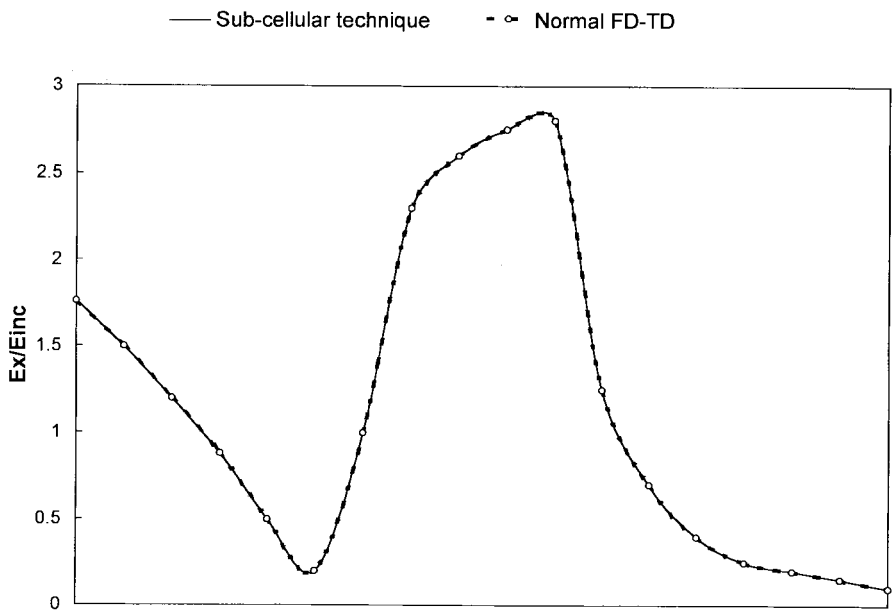

Fig. 3. Comparison of the sub-cellular technique and FDTD normal method for the magnitude of the gap electric field E_x along an observation line perpendicular to the screen and centered in the slot gap.

of wave equations which leads to some reflection. To minimize the error introduced due to it, the computational domain should be considered as large as possible and the time domain should be considered as small as possible (which can be achieved by stopping the computation as soon as the useful information has been obtained).

III. Results and Discussion

To check the validity of the sub-cellular FDTD technique, a small region of space is selected. The space is then divided into two sub-regions, one with bigger cells (coarse grid with cell size $= \lambda/20$) and the other with smaller cells (fine grid with cell size $= \lambda/60$). The space is terminated at both ends with third-order Liao absorbing boundary condition [22]. At time $= 0$, it is assumed that all the fields within the numerical sampling region are identically zero. An excitation pulse in the shape of a half-sine wave is assumed to enter the sampling region. Propagation of the pulse is modeled by the commencement of time-stepping, which is simply the implementation of the finite-difference analog of the curl equations. Time-stepping continues as the numerical analog of the pulse strikes and gets completely absorbed by the absorbing boundary at the end of the selected small region of space. The movement of the wave is then observed, using the data stored at different time-steps and animating the plots. It is found that there is negligible reflection at the boundary between the two different cell sizes, though it is not the *sufficient* condition, but an *essential* condition to support the validity of this technique.

The validity and accuracy of this sub-cellular technique in the FDTD method is further investigated and compared with the normal FDTD method for a thin slot in a conducting screen (see Fig. 2). Comparison has been done for the computed electric-field distribution. The screen is assumed to be 0.1-λ thick extending 0.5λ to each side of the slot with a gap distance of 0.025λ. Broadside TE illumination has been considered. In the case of sub-cellular technique, the coarse FDTD grid has a cell size of $\lambda/40$ while the fine grid has a cell size of $\lambda/120$ to treat the slot as a three-cell gap. The coarse- and fine-grid boundary lies at one big cell size (three small cell size) away from the slot boundary on both the sides of the slot as shown in Fig. 2. In the case of a conventional FDTD method, the cell size of $\lambda/120$ has been considered throughout.

Fig. 3 shows the magnitude of the gap electric field E_x along an observation line perpendicular to the screen and centered in the slot gap. Results show that the sub-cellular technique has excellent agreement with the conventional FDTD method. Computation time saved in the above case using this sub-cellular technique is more than 30%, as compared with the conventional FDTD method. In those cases, where the important structural feature is very small compared to the main body, the time saving can be more. As this technique does not require any modifications in the field equations or any other limitations, it can be applied to all those cases where the normal FDTD method is applicable.

IV. Conclusion

A new sub-cellular technique has been proposed for the analysis of structures which generally has important small structural features compared to the main body, in at least one dimension. This technique is very simple to implement in the FDTD method and saves considerable computation time by reducing the number of cells keeping the same accuracy. This technique is applicable to all those cases where the conventional FDTD method is applicable.

Appendix

Let $\boldsymbol{M}$ coarse-grid cells $= \boldsymbol{N}$ fine-grid cells which will be referred as the $\boldsymbol{M} : \boldsymbol{N}$ case. For the $\boldsymbol{M} : \boldsymbol{N}$ case, the general

formula for (1) will be

$$h = \boldsymbol{u} H_c + \boldsymbol{v} H_b$$

where the weighting coefficients $\boldsymbol{u}$ and $\boldsymbol{v}$ can be written as

$$\boldsymbol{u} = \frac{\boldsymbol{R}}{\boldsymbol{D}}$$
$$\boldsymbol{v} = \frac{\boldsymbol{R}+\mathbf{1}}{\mathbf{2}\boldsymbol{D}}$$
$$\boldsymbol{R} = \frac{\boldsymbol{M}}{\boldsymbol{N}}$$
$$\boldsymbol{D} = \frac{\mathbf{3}\boldsymbol{R}+\mathbf{1}}{\mathbf{2}}.$$

Similarly, for the $\boldsymbol{M} : \boldsymbol{N}$ case, the general formula for (2) will be

$$e = \boldsymbol{s} E_c + \boldsymbol{t} E_f$$

where the weighting coefficients $\boldsymbol{s}$ and $\boldsymbol{t}$ can be written as

$$s = \frac{\boldsymbol{M}-\boldsymbol{N}}{\boldsymbol{M}+\mathbf{3}\boldsymbol{N}}$$
$$\boldsymbol{t} = \frac{\mathbf{4}\boldsymbol{N}}{\boldsymbol{M}+\mathbf{3N}}.$$

ACKNOWLEDGMENT

The author would like to thank the reviewers and Dr. J. B. Schneider, Assistant Prof., Washington State University, for their valuable suggestions.

REFERENCES

[1] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. AP-14, pp. 302–307, May 1966.

[2] A. Taflove and M. E. Brodwin, "Numerical solution of steady state electromagnetic scattering problems using the time-dependent Maxwell's equations," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 623–630, Aug. 1975.

[3] ——, "Computation of the electromagnetic fields and induced temperatures within a model of the microwave-irradiated human eye," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 888–896, Nov. 1975.

[4] A. Taflove, "Review of the formulation and applications of the finite-difference time-domain method for numerical modeling of the electromagnetic wave interactions with arbitrary structures," *Wave Motion*, vol. 10, pp. 547–582, 1988.

[5] S. Kapoor and J. B. Schneider, "Characterization of microstrip discontinuities using conformal mapping and the finite difference time-domain method," *IEEE Trans. Microwave Theory Tech.*, vol. 43, pp. 2636–2639, Nov. 1990.

[6] D. M. Sheen, S. M. Ali, M. D. Abouzahra, and J. A. Kong, "Application of the three-dimensional finite-difference time-domain method to the analysis of planar microstrip circuits," *IEEE Trans. Microwave Theory Tech.*, vol. 38, pp. 849–857, July 1990.

[7] X. Zhang and K. K. Mei, "Time-domain finite-difference approach to the calculation of the frequency-dependent characteristics of microstrip discontinuities," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1775–1787, Dec. 1988.

[8] J. F. Lee, R. Palandech, and R. Mittra, "Modeling three-dimensional discontinuities in waveguides using nonorthogonal FDTD algorithm," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 346–352, Feb. 1992.

[9] J. G. Maloney and G. S. Smith, "Optimization of a conical antenna for pulse radiation: An efficient design using resistive loading," *IEEE Trans. Antennas Propagat.*, vol. 41, pp. 940-947, July 1993.

[10] G. Mur, "Absorbing boundary conditions for the finite difference approximation of the time-domain electromagnetic field equations," *IEEE Trans. Electromag. Compat.*, vol. EMC-23, pp. 377–382, Nov. 1981.

[11] K. K. Mei and J. Fang, "Superabsorption—A method to improve absorbing boundary conditions," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 1001–1010, Sept. 1992.

[12] J. G. Maloney and G. S. Smith, "The use of surface impedance concept in the FDTD method," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 38–48, Jan. 1992.

[13] J. H. Beggs, R. J. Luebbers, K. S. Yee, and K. S. Kunz, "FDTD implementation of surface impedance boundary conditions," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 49–56, Jan. 1992.

[14] K. Umashankar, A. Taflove, and B. Bekar, "Calculation and experimental validation of induced currents on coupled wires in an arbitrary shaped cavity," *IEEE Trans. Antennas Propagat.*, vol. AP-35, pp. 1248–1257, Nov. 1987.

[15] A. Taflove, K. R. Umashankar, B. Beker, F. Harfoush, and K. S. Yee, "Detailed FDTD analysis of electromagnetic fields penetrating narrow slots and lapped joints in thick conducting screens," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 247–257, Feb. 1988.

[16] K. S. Kunz and R. J. Luebbers, *The Finite Difference Time Domain Method for Electromagnetics.* Boca Raton, FL: CRC Press, 1993.

[17] K. S. Kunz and J. F. Prewitt, "Practical limitations to a natural mode characterization of electromagnetic transient response measurements," *IEEE Trans. Antennas Propagat.*, vol. AP-28, p. 575-577, July 1980.

[18] T. G. Jurgens, A. Taflove, K. Umashankar, and T. G. Moore, "Finite-difference time-domain modeling of curved surfaces," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 357–366, Apr. 1992.

[19] J. G. Maloney and G. S. Smith, "The efficient modeling of thin material sheets in the finite-difference time-domain method," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 323–330, Mar. 1992.

[20] S. S. Zivanovic, K. S. Yee, and K. K. Mei, "A subgridding method for the time-domain finite-difference method to solve Maxwell's equations," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 471–479, Mar. 1991.

[21] D. T. Prescott and N. V. Shuley, "A method for incorporating different sized cells into the finite-difference time-domain analysis technique," *IEEE Microwave Guided Wave Lett.*, vol. 2, pp. 434–436, Nov. 1992.

[22] Z. P. Liao, H. L. Wong, B. P. Yang, and Y. F. Yuan, "A transmitting boundary for transient wave analysis," *Sci. Sin., Ser. A*, vol. 27, no. 10, pp. 1063–1076, 1984.

Sunil Kapoor (S'91–M'95) received the M.Sc. degree in physics and the M.Tech. degree in microwave electronics from Delhi University, India, in 1985 and 1987, respectively, and the M.S.E.E. degree from Oregon State University, Corvallis, in 1992.

He worked as a Scientist at the Central Electronics Engineering Research Institute, India, from 1987 to 1990, and 1992 to 1993, researching broadbanding techniques of helix TWT's and designing electron guns and collectors for microwave tubes. From 1990 to 1992, he was at Oregon State University, as a Graduate Assistant, working on numerical techniques for the studies relating to microstrip discontinuities. In 1993, he joined Washington State University, Pullman, as a Research Assistant, working on the application of FDTD technique for the design of microwave devices. Since 1994, he has been with Ward Products Corporation, New Brunswick, NJ, researching active and passive antennas. He has authored or co-authored over 12 papers and has applied for two patents.

Mr. Kapoor was awarded the J. C. Bose Memorial Award, for best engineering science paper, in 1992.